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STRESSES NEAR A FLAT INCLUSION IN BONDED DISSIMILAR MATERIALS†

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Abstract—The plane elastostatic problem for bonded materials containing a flat inclusion is considered. It is assumed that the inclusion is located parallel to or on the interface and may be rigid or elastic with negligible bending rigidity. The integral equations for various cases are derived and their solutions are described. The stress state around the singular points are investigated and a pair of stress intensity factors similar to that for crack problems are defined. A series of numerical examples for two bonded half planes and for a half plane is worked out. The stress intensity factors are presented as functions of the ratio of the distance from the interface to the length of the inclusion.

1. INTRODUCTION

IN STUDYING the fracture of multi-phase materials and structures composed of bonded dissimilar solids, it is important to evaluate the stresses around imperfections such as cracks and inclusions. The edges of these imperfections are lines of stress singularity. Hence, they are expected to be the locations around which the fracture of the medium would generally nucleate. The stress distribution around a crack lying parallel to and on the interface of bonded dissimilar materials was considered in a previous series of papers [1–6].

In this paper we consider the plane elastostatic problem for an inclusion located parallel to the interface of two bonded half planes. The problem of interface inclusion for two bonded materials and the cover plate problem for a half plane are recovered as the limiting cases of the general problem. It will be assumed that the thickness of the inclusion is very small compared to its lateral dimensions. Thus, analytically, it can be approximated by a singular surface across which the displacement vector is continuous and the stress vector suffers a discontinuity. The problem will be solved for three types of inclusions. First, it will be assumed that the inclusion is completely rigid. In this case both normal and shear stresses across the inclusion will be discontinuous. In the second, we will assume that the inclusion is an inextensible surface with zero bending rigidity. Here the normal stress will be continuous and the shear stress will have a discontinuity. Finally, as a third model we will assume that the inclusion is an elastic sheet with, again, negligible bending rigidity. In this case too, the normal stress will be continuous, the shear stress will have a discontinuity, and the shear stress difference will act as a body force on the elastic sheet.

In this study we are mainly interested in evaluating the disturbance in the stress state resulting from the existence of the inclusion in the medium. Hence, it will be assumed that

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the displacement field in the bonded materials without the inclusion is known, and the only "external loads" in the disturbance problem are the displacements at the presumed location of the inclusion. Referring to Fig. 1, if $u_0(x, y)$, $v_0(x, y)$ are the displacements in material 2 for the inclusion-free medium under the given external loads, the input functions for the disturbance problem will be

$$\begin{aligned} u_2(x, 0) - u_4(x, 0) &= -u_0(x, 0) \\ v_2(x, 0) - v_4(x, 0) &= -v_0(x, 0), (-a < x < a) \end{aligned} \tag{1.1}$$

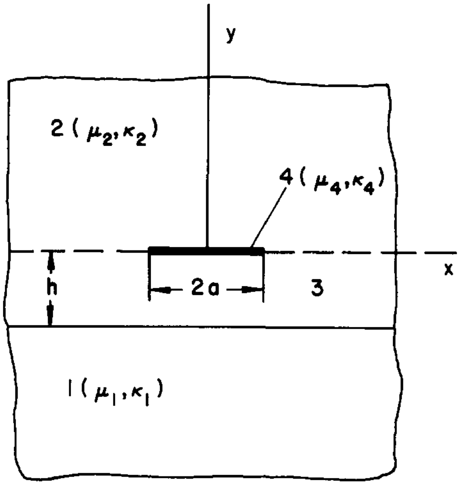

FIG. 1. The geometry of the bonded medium containing an inclusion.

where, the subscripts 2 and 4 refer to the material 2 and the inclusion 4, respectively.

In this problem we will further assume that the length of the inclusion, $2a$, and its distance from the interface, h, are sufficiently small compared to other planar dimensions of the bonded planes, so that in formulating the disturbance problem the media 1 and 2 may be treated as being semi-infinite. Since $x = 0$ is the plane of geometric symmetry, the general problem may be studied as the sum of a "symmetric" and an "anti-symmetric" problem by simply decomposing the external loads into their symmetric and anti-symmetric components. In this paper we will consider only the symmetric problem. The anti-symmetric part of the problem may be solved by following a similar procedure, resulting in only minor changes in the kernels of the integral equations.

2. FORMULATION OF THE PROBLEM

Consider the plane elasticity problem for the bonded materials containing an inclusion shown in Fig. 1. For the purpose of formulating the problem, the part of the half plane 2, $-h < y < 0$ will be treated as a separate layer, 3. Let $u_i, v_i, (i = 1, \ldots, 4)$ refer to displacement components in materials $1, \ldots, 4$ and similarly the subscripts $i = 1, \ldots, 4$ in $\sigma_{ixx}, \ldots$ refer to the stress components in various media. The problem will be solved for the external loads given by (1.1). In the symmetric problem $u_k(x, y) = -u_k(-x, y)$, $v_k(x, y) = v_k(-x, y)$,

($k = 0, 1, \ldots, 4$). Hence, for the stress disturbance problem the displacements and stresses may be expressed by the following Fourier integrals [1]:

$$u_k(x, y) = \frac{2}{\pi}\int_0^\infty [(A_{k1}+yA_{k2})\,e^{-\alpha y}+(A_{k3}+yA_{k4})\,e^{\alpha y}]\sin\alpha x\,d\alpha \tag{2.1}$$

$$v_k(x, y) = \frac{2}{\pi}\int_0^\infty \left[\left(A_{k1}+\left(\frac{\kappa_k}{\alpha}+y\right)A_{k2}\right)e^{-\alpha y}+\left(-A_{k3}+\left(\frac{\kappa_k}{\alpha}-y\right)A_{k4}\right)e^{\alpha y}\right]\cos\alpha x\,d\alpha \tag{2.2}$$

$$\frac{1}{2\mu_k}\sigma_{kyy} = \frac{2}{\pi}\int_0^\infty \left[-\left(\alpha(A_{k1}+A_{k2}y)+\frac{1+\kappa_k}{2}A_{k2}\right)e^{-\alpha y} + \left(-\alpha(A_{k3}+A_{ky}y)+\frac{1+\kappa_k}{2}A_{k4}\right)e^{\alpha y}\right]\cos\alpha x\,d\alpha \tag{2.3}$$

$$\frac{1}{2\mu_k}\sigma_{kxy} = \frac{2}{\pi}\int_0^\infty \left[-\left(\alpha(A_{k1}+A_{k2}y)+\frac{\kappa_k-1}{2}A_{k2}\right)e^{-\alpha y} + \left(\alpha(A_{k3}+A_{k4}y)-\frac{\kappa_k-1}{2}A_{k4}\right)e^{\alpha y}\right]\sin\alpha x\,d\alpha \tag{2.4}$$

$$\frac{1}{2\mu_k}\sigma_{kxx} = \frac{2}{\pi}\int_0^\infty \left[\left(\alpha(A_{k1}+A_{k2}y)-\frac{3-\kappa_k}{2}A_{k2}\right)e^{-\alpha y} + \left(\alpha(A_{k3}+A_{k4}y)+\frac{3-\kappa_k}{2}A_{k4}\right)e^{\alpha y}\right]\cos\alpha x\,d\alpha \tag{2.5}$$

where $A_{kj}(\alpha)$, ($k = 1, 2, 3$; $j = 1, \ldots, 4$) are unknown functions of α and $\kappa_k = 3-4\nu_k$ for plane strain and $\kappa_k = (3-\nu_k)/(1+\nu_k)$ for generalized plane stress, ν_k, μ_k being the elastic constants. The conditions at $y = \mp\infty$ require that

$$A_{11} = 0 = A_{12}, \qquad A_{23} = 0 = A_{24}. \tag{2.6}$$

At $y = 0$ and $y = -h$ we have the following continuity conditions:

$$u_2(x, 0) = u_3(x, 0), \qquad v_2(x, 0) = v_3(x, 0) \tag{2.7}$$

$$u_3(x, -h) = u_1(x, -h), \qquad v_3(x, -h) = v_1(x, -h) \tag{2.8}$$

$$\sigma_{3yy}(x, -h) = \sigma_{1yy}(x, -h), \qquad \sigma_{3xy}(x, -h) = \sigma_{1xy}(x, -h). \tag{2.9}$$

By using the conditions (2.6)–(2.9), ten of the twelve functions A_{kj} appearing in (2.1)–(2.5) can be eliminated. The remaining two constants are determined from the mixed boundary conditions at $y = 0$. These are the statements of the physical conditions that on $y = 0$, $|x| < a$ the displacements are specified and on $y = 0$, $|x| > a$ the stress vector is continuous. Thus if we define

$$f_1(x) = -\frac{\partial}{\partial x}u_0(x, 0), \qquad f_2(x) = -\frac{\partial}{\partial x}v_0(x, 0), \qquad |x| < a. \tag{2.10}$$

The conditions at $y = 0$ may be expressed as

$$\lim_{y\to 0+}\left[\frac{\partial}{\partial x}u_2(x,y)-\frac{\partial}{\partial x}u_4(x)\right] = f_1(x), \qquad |x| < a$$
$$\lim_{y\to 0+}\left[\frac{\partial}{\partial x}v_2(x,y)-\frac{\partial}{\partial x}v_4(x)\right] = f_2(x), \qquad |x| < a \tag{2.11}$$

$$\sigma_{2yy}(x,0)-\sigma_{3yy}(x,0) = 0, \qquad |x| > a$$
$$\sigma_{2xy}(x,0)-\sigma_{3xy}(x,0) = 0, \qquad |x| > a. \tag{2.12}$$

In the three types of inclusions mentioned in Section 1, conditions (2.11) and (2.12) may further be reduced as follows:

(a) Rigid inclusion. In this case $\partial u_4/\partial x = 0$, $\partial v_4/\partial x = 0$ and (2.11) becomes

$$\lim_{y\to 0+}\frac{\partial}{\partial x}u_2(x,y) = f_1(x) = f_1(-x), \qquad |x| < a$$
$$\lim_{y\to 0+}\frac{\partial}{\partial x}v_2(x,y) = f_2(x) = -f_2(-x), \qquad |x| < a \tag{2.13}$$

whereas (2.12) remains the same.

(b) Flexible, inextensible inclusion. Since the bending rigidity of the inclusion is zero, in this case (2.11) and (2.12) may be simplified as follows:

$$\sigma_{2yy}(x,0) = \sigma_{3yy}(x,0), \qquad -\infty < x < \infty, \tag{2.14}$$

$$\lim_{y\to 0+}\frac{\partial}{\partial x}u_2(x,y) = f_1(x), \qquad |x| < a \tag{2.15}$$

$$\sigma_{2xy}(x,0)-\sigma_{3xy}(x,0) = 0, \qquad |x| > a \tag{2.16}$$

(c) Flexible, elastic inclusion. In this case too, the bending rigidity of the inclusion is zero and the conditions at $y = 0$ become

$$\sigma_{2yy}(x,0) = \sigma_{3yy}(x,0), \qquad -\infty < x < \infty, \tag{2.17}$$

$$\lim_{y\to 0+}\left[\frac{\partial}{\partial x}u_2(x,y)-\frac{\partial}{\partial x}u_4(x)\right] = f_1(x), \qquad |x| < a \tag{2.18}$$

$$\sigma_{2xy}(x,0)-\sigma_{3xy}(x,0) = 0, \qquad |x| > a. \tag{2.19}$$

Note that in the problems (b) and (c) (2.14) and (2.17), respectively, provide another algebraic expression in $A_{kj}(\alpha)$ and only one unknown function has to be determined from the mixed boundary conditions. Also note that (b) is a special case of (c) in which $\mu_4 = \infty$. Thus, in deriving the integral equations only the cases (a) and (c) will be considered.

(a) *The integral equations for the rigid inclusion*

After eliminating ten of the functions A_{kj} by using (2.6)–(2.9), the mixed boundary conditions (2.12) and (2.13) may be expressed as

$$
\begin{aligned}
(\sigma_{2yy}-\sigma_{3yy})_{y=0} &= \frac{2}{\pi}\int_0^\infty \frac{\mu_2(1+\kappa_2)}{\kappa_2}(2\alpha B_3-2\kappa_2 B_4)\cos\alpha x\,\mathrm{d}\alpha = 0,\\
(\sigma_{2xy}-\sigma_{3xy})_{y=0} &= \frac{2}{\pi}\int_0^\infty -\frac{\mu_2(1+\kappa_2)}{\kappa_2}2\alpha B_3\sin\alpha x\,\mathrm{d}\alpha = 0, \qquad |x|>a,
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\lim_{y\to 0+}\frac{\partial u_2}{\partial x} &= \lim_{y\to 0+}\frac{2}{\pi}\int_0^\infty \alpha(B_1+B_3)\,\mathrm{e}^{-\alpha y}\cos\alpha x\,\mathrm{d}\alpha = f_1(x),\\
\lim_{y\to 0+}\frac{\partial v_2}{\partial x} &= \lim_{y\to 0+}\frac{2}{\pi}\int_0^\infty [\alpha(B_1-B_3)+\kappa_2(B_2+B_4)]\,\mathrm{e}^{-\alpha y}\sin\alpha x\,\mathrm{d}\alpha = f_2(x), \qquad |x|<a,
\end{aligned}
\tag{2.21}
$$

where

$$
\begin{aligned}
2\alpha B_1 &= \left[B_4\left(\frac{b_1}{b_2}+\frac{4\alpha^2h^2-\kappa_2^2}{1-\kappa_1(b_1-b_2)}\right)+2\alpha B_3\frac{2\alpha h-\kappa_2}{\kappa_1(b_1-b_2)-1}\right]\mathrm{e}^{-2\alpha h},\\
B_2 &= \frac{2\alpha B_3-B_4(2\alpha h+\kappa_2)}{\kappa_1(b_1-b_2)-1}\mathrm{e}^{-2\alpha h},\\
b_1 &= \frac{\kappa_1\mu_2-\kappa_2\mu_1}{\kappa_1(\mu_2-\mu_1)}, \qquad b_2 = \frac{\mu_1+\kappa_1\mu_2}{\kappa_1(\mu_2-\mu_1)}.
\end{aligned}
\tag{2.22}
$$

Thus, in the system of dual integral equations (2.20) and (2.21) there are only two unknown functions, $B_3(\alpha)$ and $B_4(\alpha)$.

We will now define the following two new unknown functions:

$$
\begin{aligned}
\sigma_{2yy}(x,0)-\sigma_{3yy}(x,0) &= p_2(x),\\
\sigma_{2xy}(x,0)-\sigma_{3xy}(x,0) &= p_1(x)
\end{aligned}
\tag{2.23}
$$

and note that $p_1(x) = 0 = p_2(x)$ for $|x| > a$. From (2.20) and (2.23) we obtain

$$
\begin{aligned}
A(\alpha) &= \frac{\mu_2(1+\kappa_2)}{\kappa_2}(2\alpha B_3-2\kappa_2 B_4) = \int_0^a p_2(t)\cos\alpha t\,\mathrm{d}t,\\
B(\alpha) &= -\frac{\mu_2(1+\kappa_2)}{\kappa_2}2\alpha B_3 = \int_0^a p_1(t)\sin\alpha t\,\mathrm{d}t.
\end{aligned}
\tag{2.24}
$$

After normalizing the distances with respect to the half length of the inclusion, a, and using the relation

$$
\lim_{y\to 0+}\int_{-1}^{1} p(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{-\alpha y}\sin\alpha(t-x)\,\mathrm{d}\alpha = \int_{-1}^{1}\frac{p(t)\,\mathrm{d}t}{t-x}
\tag{2.25}
$$

from (2.21), (2.22) and (2.24) we obtain

$$
\frac{1}{\pi}\int_{-1}^{1}\frac{p_k(t)}{t-x}\,\mathrm{d}t+\int_{-1}^{1}\sum_{1}^{2}k_{kj}(t,x)p_j(t)\,\mathrm{d}t = -\frac{2\mu_2(1+\kappa_2)}{\kappa_2}f_k(x), \qquad |x|<1, \qquad k=1,2
\tag{2.26}
$$

where

$$\pi k_{11}(t,x) = \frac{I_1}{2\kappa_2}\left(\frac{b_1}{b_2}+\frac{\kappa_2^2}{1-\kappa_1(b_1-b_2)}\right)+\frac{2h(\kappa_2 I_3+hI_4)}{\kappa_2(1-\kappa_1(b_1-b_2))},$$

$$\pi k_{12} = \pi k_{21} = \frac{I_2}{2\kappa_2}\left(\frac{b_1}{b_2}-\frac{\kappa_2^2}{1-\kappa_1(b_1-b_2)}\right)+\frac{2h^2 I_5}{\kappa_2(1-\kappa_1(b_1-b_2))},$$

$$\pi k_{22} = \frac{I_1}{2\kappa_2}\left(\frac{b_1}{b_2}+\frac{\kappa_2^2}{1-\kappa_1(b_1-b_2)}\right)-\frac{2h(\kappa_2 I_3-hI_4)}{\kappa_2(1-\kappa_1(b_1-b_2))}, \tag{2.27}$$

$$I_1 = \frac{t-x}{4h^2+(t-x)^2}, \qquad I_2 = \frac{2h}{4h^2+(t-x)^2}, \qquad I_3 = 2I_1 I_2,$$

$$I_4 = 2I_1(3I_2^2-I_1^2), \qquad I_5 = 2I_2(I_2^2-3I_1^2)$$

and h stands for h/a. The static equilibrium of the inclusion requires that

$$\int_{-1}^{1} p_k(t)\,\mathrm{d}t = 0, \qquad k = 1, 2. \tag{2.28}$$

Thus, (2.26) will be solved subject to the conditions (2.28).

For $h \to \infty$, $k_{ij} \to 0$ and (2.26) reduces to a pair of uncoupled equations for the plane with a rigid line inclusion.

In another special case when $h \to 0$, the expressions for B_1 and B_2 given by (2.22) become

$$2\alpha B_1 = \left(\frac{b_1}{b_2}-\frac{\kappa_2^2}{1-\kappa_1(b_1-b_2)}\right)B_4+\frac{2\alpha B_3\kappa_2}{1-\kappa_1(b_1-b_2)}$$

$$B_2 = \frac{\kappa_2 B_4-2\alpha B_3}{1-\kappa_1(b_1-b_2)}. \tag{2.29}$$

Taking this into account, using the relation

$$\lim_{y\to 0+}\int_{-1}^{1} p(t)\,\mathrm{d}t\int_0^\infty \mathrm{e}^{-\alpha y}\cos\alpha(t-x)\,\mathrm{d}\alpha = \pi p(x) \tag{2.30}$$

and defining the new unknown functions p_1 and p_2 as

$$\sigma_{2yy}(x,0)-\sigma_{1yy}(x,0) = p_2(x)$$

$$\sigma_{2xy}(x,0)-\sigma_{1xy}(x,0) = p_1(x) \tag{2.31}$$

from (2.21) we obtain

$$-\frac{1}{\pi}\int_{-1}^{1}\frac{p_1(t)}{t-x}\,\mathrm{d}t+\gamma p_2(x) = \frac{4\mu_2(1+\kappa_2)\gamma}{c}f_1(x), \qquad |x|<1$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p_2(t)}{t-x}\,\mathrm{d}t+\gamma p_1(x) = \frac{4\mu_2(1+\kappa_2)\gamma}{c}f_2(x), \qquad |x|<1 \tag{2.32}$$

where

$$\gamma = \frac{\kappa_1(\mu_2+\kappa_2\mu_1)-\kappa_2(\mu_1+\kappa_1\mu_2)}{\kappa_1(\mu_2+\kappa_2\mu_1)+\kappa_2(\mu_1+\kappa_1\mu_2)}$$

$$c = \frac{\kappa_1\mu_2-\kappa_2\mu_1}{\mu_1+\kappa_1\mu_2}+\frac{(\mu_1-\mu_2)\kappa_2^2}{\mu_2+\kappa_2\mu_1}.$$

(b) *The integral equation for the elastic inclusion*

For this problem the mixed boundary conditions are given by (2.18) and (2.19). If we define a new unknown function $p(x)$ by

$$\sigma_{2xy}(x,0)-\sigma_{3xy}(x,0) = p(x) \tag{2.33}$$

from the static equilibrium of the inclusion we obtain

$$\frac{\partial u_4}{\partial x} = -\frac{1+\kappa_4}{8\mu_4 h_4}\int_{-a}^{x} p(t)\,\mathrm{d}t \tag{2.34}$$

where, h_4 is the thickness and μ_4, κ_4 are the elastic constants of the inclusion. From the basic derivation (2.20) and (2.21), taking into account (2.17)–(2.19) and (2.34), the dual integral equations for this problem may be obtained as

$$\frac{2}{\pi}\int_0^\infty \frac{\mu_2(1+\kappa_2)}{\kappa_2}2\alpha B_3 \sin\alpha x\,\mathrm{d}\alpha = 0, \qquad |x| > a \tag{2.35}$$

$$\lim_{y\to 0+}\frac{2}{\pi}\int_0^\infty \alpha(B_1+B_3)\,\mathrm{e}^{-\alpha y}\cos\alpha x\,\mathrm{d}\alpha+\frac{1+\kappa_4}{8\mu_4 h_4}\int_{-a}^{x} p(t)\,\mathrm{d}t = f_1(x), \qquad |x| < a \tag{2.36}$$

with $\kappa_2 B_4 = \alpha B_3$ and B_1, B_2 as given by (2.22). From (2.29) and (2.35) we have

$$C(\alpha) = \frac{\mu_2(1+\kappa_2)}{\kappa_2}2\alpha B_3 = \int_0^a p(t)\sin\alpha t\,\mathrm{d}t. \tag{2.37}$$

Substituting this into (2.36), using (2.25), and again normalizing with respect to a, we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p(t)}{t-x}\mathrm{d}t+\int_{-1}^{1}k(x,t)p(t)\,\mathrm{d}t-\frac{(1+\kappa_2)^2}{2\kappa_2}\lambda\int_{-1}^{x}p(t)\,\mathrm{d}t = -\frac{2\mu_2(1+\kappa_2)}{\kappa_2}f_1(x), \qquad |x| < a \tag{2.38}$$

where

$$k(x,t) = \frac{1}{\pi}\int_0^\infty \frac{1}{2\kappa_2}\left[\frac{b_1}{b_2}+\frac{(2\alpha h-\kappa_2)^2}{1-\kappa_1(b_1-b_2)}\right]\mathrm{e}^{-2\alpha h}\sin\alpha(t-x)\,\mathrm{d}\alpha,$$

$$\lambda = \frac{1+\kappa_4}{1+\kappa_2}\frac{\mu_2}{2\mu_4 h_4}. \tag{2.39}$$

Note that the integral equation for the inextensible inclusion is obtained by simply letting $\lambda = 0$ in (2.38).

In the special case when $h \to 0$, $k(x, t)$ becomes a simple Cauchy kernel [see, (2.25)] and (2.38) reduces to

$$\left[1+\frac{1}{2\kappa_2}\left(\frac{b_1}{b_2}+\frac{\kappa_2^2}{1-\kappa_1(b_1-b_2)}\right)\right]\frac{1}{\pi}\int_{-1}^{1}\frac{p(t)\,dt}{t-x}-\frac{(1+\kappa_2)^2}{2\kappa_2}\lambda\int_{-1}^{x}p(t)\,dt$$
$$= -\frac{2\mu_2(1+\kappa_2)}{\kappa_2}f_1(x), \qquad |x| < 1. \tag{2.40}$$

If we now let $\mu_1 = 0$, we have $b_1 = b_2 = 1$ and (2.40) becomes

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p(t)}{t-x}\,dt-\lambda\int_{-1}^{x}p(t)\,dt = -\frac{4\mu_2}{1+\kappa_2}f_1(x), \qquad |x| < 1 \tag{2.41}$$

which is the integral equation for the problem of elastic cover plate [7]. Equations (2.38), (2.40) and (2.41) must, again, be solved under the following equilibrium condition:

$$\int_{-1}^{1} p(x)\,dx = 0. \tag{2.42}$$

3. SOLUTION OF THE INTEGRAL EQUATIONS

To solve the system of singular integral equations (2.26), the method described in [8] will be used. Around the singular points ∓ 1, the unknown functions p_k have integrable singularities. Thus, by following the procedure of [9], from the examination of the dominant part of the system (2.26) it can be shown that the fundamental function of the integral equations is

$$w(x) = (1-x^2)^{-\frac{1}{2}}. \tag{3.1}$$

Observing that this is the weight function of the Chebyshev polynomials $T_n(x)$, the singularities of the integral equations (2.26) may be removed by defining the unknown functions as follows:

$$p_1(x) = w(x)\sum_{1}^{\infty} a_n T_{2n-1}(x)$$
$$p_2(x) = w(x)\sum_{0}^{\infty} b_n T_{2n}(x) \tag{3.2}$$

where a_n, b_n are unknown constants and the symmetry conditions $p_1(x) = -p_1(-x)$, $p_2(x) = p_2(-x)$ have been taken into account. From the equilibrium conditions (2.28) and the orthogonality of $T_n(x)$, it is clear that $b_0 = 0$. To determine the constants a_n, b_n $(n = 1, 2, \ldots)$, we substitute (3.2) into (2.26) and use the following relation [10]

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(t)\,dt}{(t-x)(1-t^2)^{\frac{1}{2}}} = \begin{cases} 0 & , \quad n = 0 \\ U_{n-1}(x), & \quad n > 0, \end{cases} \qquad |x| < 1$$

to remove the singularities. In (3.3) $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind. Equation (2.26) then becomes

$$\begin{aligned}\sum_1^\infty [a_n U_{2n-2}(x)+a_n H^{11}_{2n-1}(x)+b_n H^{12}_{2n}] &= -\frac{2\mu_2(1+\kappa_2)}{\kappa_2} f_1(x)\\ \sum_1^\infty [b_n U_{2n-1}(x)+a_n H^{21}_{2n-1}(x)+b_n H^{22}_{2n}(x)] &= -\frac{2\mu_2(1+\kappa_2)}{\kappa_2} f_2(x), \qquad |x|<1\end{aligned} \tag{3.4}$$

where

$$H^{kj}_m(x) = \int_{-1}^{1} k_{kj}(x,t)\frac{T_m(t)}{(1-t^2)^{\frac{1}{2}}}\,\mathrm{d}t. \tag{3.5}$$

Equations (3.4) are further reduced to a system of linear algebraic equations in a_n, b_n by multiplying the first equations by U_{2k-2} and the second equation by U_{2k-1}, ($k = 1, 2, \ldots$) and integrating in $(-1, 1)$. This system of equations may then be solved by the method of reduction [11].

In a similar way, the solution of the integral equations (2.38), (2.40) and (2.41) may be obtained in the following form:

$$p(x) = (1-x^2)^{-\frac{1}{2}} \sum_1^\infty c_n T_{2n-1}(x) \tag{3.6}$$

which satisfies the condition (2.42).

The system of integral equations (2.32) may easily be obtained in closed form by defining

$$\phi(x) = p_2(x)+ip_1(x) \tag{3.7}$$

and using the function–theoretic method outlined in [9]. From (2.32) and (3.7) we find

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{\phi(t)\,\mathrm{d}t}{t-x}-\gamma\phi(x) = g(x), \qquad |x|<1 \tag{3.8}$$

where

$$g(x) = -\frac{4\mu_2(1+\kappa_2)\gamma}{c}(f_1+if_2).$$

$\phi(x)$ may then be obtained as

$$\begin{aligned}\phi(x) &= F^+(x)-F^-(x),\\ F(z) &= R(z)\left(\frac{1}{1-\gamma}\frac{1}{2\pi i}\int_{-1}^{1}\frac{g(t)\,\mathrm{d}t}{(t-z)R^+(t)}+C\right),\\ R(z) &= (z-1)^\alpha(z+1)^\beta,\\ \alpha = -\tfrac{1}{2}-i\omega, \qquad \beta &= -\tfrac{1}{2}+i\omega, \qquad \omega = \frac{1}{2\pi}\log\left(\frac{1+\gamma}{1-\gamma}\right),\end{aligned} \tag{3.9}$$

where the constant C is obtained from the following equilibrium condition

$$\int_{-1}^{1}\phi(x)\,\mathrm{d}x = 0. \tag{3.10}$$

4. STRESS STATE AROUND THE SINGULAR POINTS

(a) *The rigid inclusion*

Even though the solution of the integral equations discussed in the previous section directly yield some important information about the problem, it is desirable to have a method of obtaining the stress state in the medium, particularly around the singular points ∓ 1. Going back to the definition of the stresses (2.3)–(2.5), (which have integrable singularities at $y = 0, x = \mp 1$), after some algebra, the dominant terms in these expressions may be obtained in terms of the functions $A(\alpha)$ and $B(\alpha)$ defined by (2.24) as follows:

$$\begin{aligned}
\sigma_{2yy}(x, y) &= \frac{1}{\pi}\int_0^\infty \left[A(\alpha)+\frac{\kappa_2-1}{\kappa_2+1}B(\alpha)\right] e^{-\alpha y}\cos\alpha x\, d\alpha + 0(r^{\frac{1}{2}}),\\
\sigma_{2xy}(x, y) &= \frac{1}{\pi}\int_0^\infty \left[B(\alpha)+\frac{\kappa_2-1}{\kappa_2+1}A(\alpha)\right] e^{-\alpha y}\sin\alpha x\, d\alpha + 0(r^{\frac{1}{2}}),\\
\sigma_{2xx}(x, y) &= \frac{1}{\pi}\int_0^\infty \left[\frac{3-\kappa_2}{1+\kappa_2}A(\alpha)-\frac{3+\kappa_2}{1+\kappa_2}B(\alpha)\right] e^{-\alpha y}\cos\alpha x\, d\alpha + 0(r^{\frac{1}{2}}), \qquad r^2 = (x-1)^2+y^2.
\end{aligned} \tag{4.1}$$

If we now substitute from (2.24) and (3.2) into (4.1) and use the following relations (see, for example, [12]),

$$\begin{aligned}
&\int_0^1 \frac{T_{2n}(t)}{(1-t^2)^{\frac{1}{2}}}\cos\alpha t\, dt = (-1)^n\frac{\pi}{2}J_{2n}(\alpha),\\
&\int_0^1 \frac{T_{2n-1}(t)}{(1-t^2)^{\frac{1}{2}}}\sin\alpha t\, dt = (-1)^{n-1}\frac{\pi}{2}J_{2n-1}(\alpha),\\
&\int_0^\infty J_{2n}(\alpha)\, e^{-\alpha y}\frac{\cos\alpha x}{\sin\alpha x}\, d\alpha = (-1)^n\int_0^\infty J_0(\alpha)\, e^{-\alpha y}\frac{\cos\alpha x}{\sin\alpha x}\, d\alpha\\
&\qquad + 0(r) = (-1)^n\frac{1}{\sqrt{(2r)}}\frac{\sin\theta/2}{\cos\theta/2} + 0(r),\\
&\int_0^\infty J_{2n-1}(\alpha)\, e^{-\alpha y}\frac{\cos\alpha x}{\sin\alpha x}\, d\alpha = (-1)^{n-1}\int_0^\infty J_1(\alpha)\, e^{-\alpha y}\frac{\cos\alpha x}{\sin\alpha x}\, d\alpha\\
&\qquad + 0(r) = \mp(-1)^{n-1}\frac{1}{\sqrt{(2r)}}\frac{\cos\theta/2}{\sin\theta/2}\\
&\qquad + 0(r), \qquad r\, e^{i\theta} = (x-1)+iy
\end{aligned} \tag{4.2}$$

the stresses around the singular point $y = 0, x = 1$ may be expressed as

$$\begin{aligned}
\sigma_{2yy}(r, \theta) &= \frac{1}{\sqrt{(2r)}}\left[k_1\cos\frac{\theta}{2}+\frac{\kappa_2+1}{\kappa_2-1}k_2\sin\frac{\theta}{2}\right] + 0(r^{\frac{1}{2}}),\\
\sigma_{2xy}(r, \theta) &= \frac{1}{\sqrt{(2r)}}\left[-\frac{\kappa_2+1}{\kappa_2-1}k_1\sin\frac{\theta}{2}+k_2\cos\frac{\theta}{2}\right] + 0(r^{\frac{1}{2}}),\\
\sigma_{2xx}(r, \theta) &= \frac{1}{\sqrt{(2r)}}\left[-\frac{3+\kappa_2}{\kappa_2-1}k_1\cos\frac{\theta}{2}+\frac{3-\kappa_2}{\kappa_2-1}k_2\sin\frac{\theta}{2}\right] + 0(r^{\frac{1}{2}})
\end{aligned} \tag{4.3}$$

where the "stress intensity factors" k_1 and k_2 are found in terms of a_n and b_n as follows:

$$\begin{aligned} k_1 &= -\frac{\kappa_2-1}{2(\kappa_2+1)}\sum_1^\infty a_n \\ k_2 &= \frac{\kappa_2-1}{2(\kappa_2+1)}\sum_1^\infty b_n. \end{aligned} \tag{4.4}$$

To evaluate the stresses and displacements elsewhere in the composite medium, one has to go back to the general expressions (2.1)–(2.5), express the functions $A_{kj}(\alpha)$ in terms of $A(\alpha)$ and $B(\alpha)$ and through (2.24), in terms of p_1 and p_2, and evaluate the integrals. The procedure is straightforward but very time-consuming. From the view point of fracture analysis, however, the knowledge of k_1 and k_2 is sufficient.

Here we note that if the inclusion is imbedded into a homogeneous medium, as seen from (4.3), the stresses have a simple $r^{-\frac{1}{2}}$ type singularity. On the other hand, the solution given by (3.9) indicates that if the rigid inclusion is on the interface of two dissimilar materials, the stress singularity is of oscillating nature, even though it has the same $-\frac{1}{2}$ power. This behavior is identical to that observed in bonded dissimilar materials containing a crack.

(b) *The elastic inclusion*

Following a procedure similar to that outlined for the rigid inclusion above, for the flexible elastic inclusion the stresses in the neighborhood of $y = 0, x = 1$ may be expressed as

$$\begin{aligned} \sigma_{2yy} &= -\frac{1}{\pi(1+\kappa_2)}\int_0^\infty C(\alpha)\,\mathrm{e}^{-\alpha y}\cos\alpha x\,\mathrm{d}\alpha + 0(\sqrt{r}), \\ \sigma_{2xy} &= \frac{1}{\pi(1+\kappa_2)}\int_0^\infty C(\alpha)\,\mathrm{e}^{-\alpha y}\sin\alpha x\,\mathrm{d}\alpha + 0(\sqrt{r}), \\ \sigma_{2xx} &= \frac{3}{\pi(1+\kappa_2)}\int_0^\infty C(\alpha)\,\mathrm{e}^{-\alpha y}\cos\alpha x\,\mathrm{d}\alpha + 0(\sqrt{r}), \\ r\,\mathrm{e}^{i\theta} &= (x-1)+iy \end{aligned} \tag{4.5}$$

where $C(\alpha)$ is given in terms of $p(t)$ by (2.37). Now, substituting from (3.6) and (2.37) into (4.5) and using (4.2), the stresses around the singular point $(1, 0)$ are obtained as follows:

$$\begin{aligned} \sigma_{2yy}(r,\theta) &= \frac{k}{\sqrt{(2r)}}\cos\frac{\theta}{2} + 0(\sqrt{r}), \\ \sigma_{2xy}(r,\theta) &= \frac{k}{\sqrt{(2r)}}\sin\frac{\theta}{2} + 0(\sqrt{r}), \\ \sigma_{2xx}(r,\theta) &= -\frac{3k}{\sqrt{(2r)}}\cos\frac{\theta}{2} + 0(\sqrt{r}), \end{aligned} \tag{4.6}$$

where the stress intensity factor, k, is found to be

$$k = \frac{1}{2(\kappa_2+1)}\sum_1^\infty c_n. \tag{4.7}$$

5. NUMERICAL RESULTS

Numerical examples for the problems discussed in this paper will be given for the following input functions:

$$\frac{\partial u_0}{\partial x} = -f_1(x) = \varepsilon_0, \qquad \frac{\partial v_0}{\partial x} = -f_2(x) = 0. \tag{5.1}$$

As output, only the stress intensity factors have been calculated. The results are shown in Figs. 2–4.

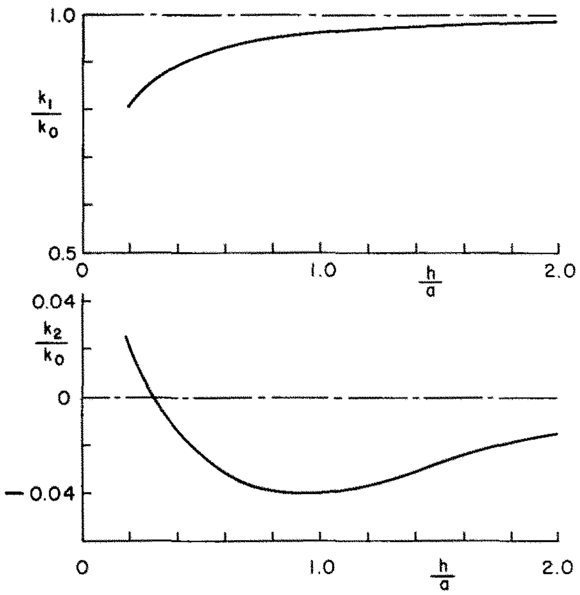

FIG. 2. The stress intensity factors for a rigid inclusion. Loading: $u_0 = \varepsilon_0 x, v_0 = 0$. Materials: 1, aluminum ($E_1 = 10^7$ psi, $\nu_1 = 0{\cdot}3$); 2, epoxy ($E_2 = 4{\cdot}5 \times 10^5$ psi, $\nu_2 = 0{\cdot}35$), $k_0 = \mu_2\varepsilon_0\sqrt{(a)}(\kappa_2-1)/\kappa_2$.

Figure 2 shows the normalized stress intensity factors $k_1/k_0, k_2/k_0$ with

$$k_0 = \mu_2\varepsilon_0\sqrt{(a)}\frac{\kappa_2-1}{\kappa_2} \tag{5.2}$$

plotted as functions of h/a, where h is the distance of the inclusion to the interface and a is the half length of the inclusion (see Fig. 1). The constants k_1 and k_2 are related to the stresses around the end points of the inclusion through the expressions given by (4.3). In this example the elastic constants of the materials are selected as $(\mu_1/\mu_2) = 23{\cdot}077$, $\nu_1 = 0{\cdot}3$, $\nu_2 = 0{\cdot}35$ (roughly aluminum and an epoxy combination). Figure 3 shows the similar results for a half-plane containing a rigid inclusion [i.e. $(\mu_1/\mu_2) = 0$, $\nu_2 = 0{\cdot}35$]. These figures indicate that as $h/a \to \infty$, the stress intensity factor ratios k_1/k_0 and k_2/k_0

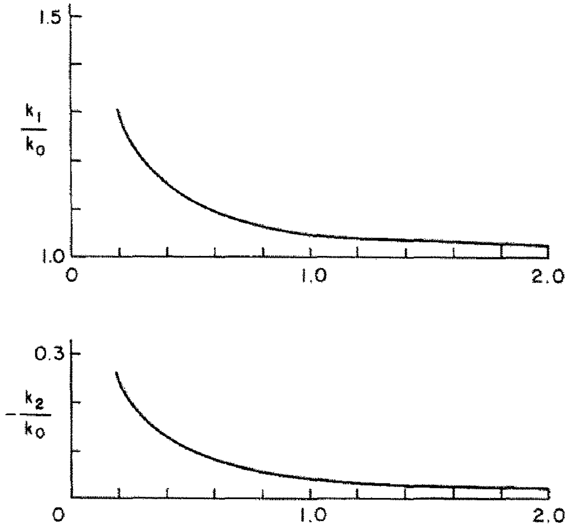

FIG. 3. The stress intensity factors for a rigid inclusion in a half plane. Loading: $u_0 = \varepsilon_0 x, v_0 = 0$. Material: $\nu = \nu_2 = 0{\cdot}35, (\mu_1/\mu_2) = 0, k_0 = \mu_2\varepsilon_0\sqrt{(a)}(\kappa_2 - 1)/\kappa_2$.

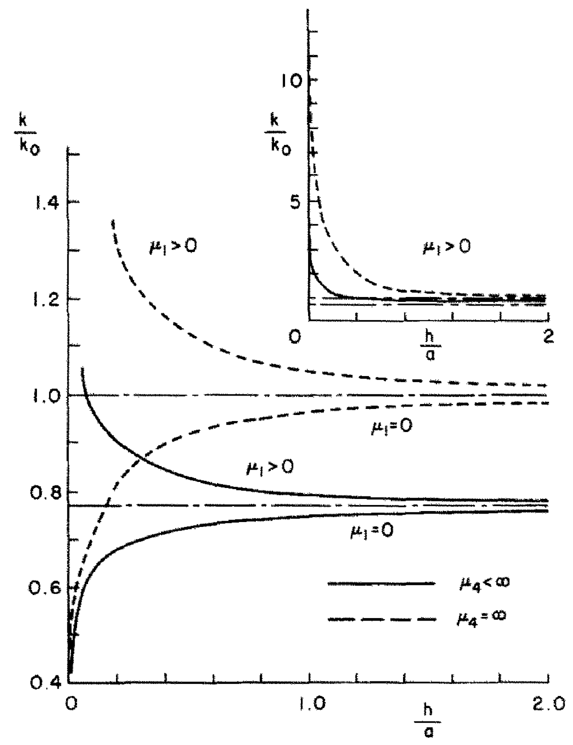

FIG. 4. The stress intensity factors for an elastic and an inextensible inclusion. Loading: $u_0 = \varepsilon_0 x$, $v_0 = 0$. Materials: 2, epoxy ($E_2 = 4{\cdot}5\times10^5$ psi, $\nu_2 = 0{\cdot}35$); 1, aluminum ($E_1 = 10^7$ psi, $\nu_1 = 0{\cdot}3$) or $(\mu_1/\mu_2) = 0$, $E_4 = 10^7$ psi, $\nu_4 = 0{\cdot}3$, $k_0 = \mu_2\varepsilon_0\sqrt{(a)}/\kappa_2$.

approach 1 and 0, respectively, which are the theoretical values for a homogeneous plane containing a rigid inclusion. As h/a decreases, the magnitude of the dominant stress intensity factor for this loading, k_1, increases for $\mu_1 > \mu_2$ and decreases for $\mu_1 < \mu_2$. This behavior which is physically expected is the opposite of that observed for the same composite medium with a crack as the imperfection.

Figure 4 shows the results for an elastic and an inextensible inclusion, both with zero bending rigidity. In this case there is only one stress intensity factor which is plotted in normalized form k/k_0 as a function of h/a, where k_0 is given by

$$k_0 = \mu_2 \varepsilon_0 \sqrt{(a)}/\kappa_2. \tag{5.3}$$

The stress state around the singular points is related to k through (4.6). From the figure it is again seen that for both the elastic and the inextensible inclusions, if $\mu_1 > \mu_2$, the stress intensity factor is greater than the respective values corresponding to the infinite homogeneous plate with a central inclusion [i.e. $(h/a) = \infty$], whereas for $(\mu_1/\mu_2) = 0$, the stress intensity factor is smaller than these asymptotic values. In the inextensible inclusion problem, for $h = \infty$ we have $k(x, t) = 0$, $\lambda = 0$ and (2.38) becomes

$$\frac{1}{\pi}\int_{-1}^{1} \frac{p(t)\,\mathrm{d}t}{t-x} = (1+\kappa_2)\frac{2\mu_2\varepsilon_0}{\kappa_2}, \qquad |x| < 1. \tag{5.4}$$

From (3.3) and (3.6) it is seen that (5.4) has the following solution:

$$p(x) = \frac{2\mu_2\varepsilon_0}{\kappa_2}(1+\kappa_2)\frac{x}{(1-x^2)^{\frac{1}{2}}}, \qquad |x| < 1 \tag{5.5}$$

from which, by (4.7) and (5.3) (and $a = 1$) the stress intensity factor ratio k/k_0 is found to be unity. The elastic inclusion problem does not have a closed form solution.

For $h = 0$ the results are obtained from the solution of (2.40). Thus, in this problem as $h \to 0$, the stress intensity factor approaches a definite limit. This is not the case in the rigid inclusion problem, in which for $h = 0$ the nature of the system of integral equations and, as a result, the behavior of the solution are different than that corresponding to $h \neq 0$. This may be seen by comparing (2.26) and (2.32) (where the former is a system of integral equations of the first kind, and the latter is of the second kind) and the solutions (3.2) and (3.9) (where the former has a simple $r^{-\frac{1}{2}}$ type singularity and the latter has an oscillating singularity).

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Абстракт—Исследуется плоская упругостатическая задача, для соединенных материалов, в которых находится плоское включение. Предполагается, что включение расположенное параллельно к границе поверхности раздела, или на границе. Оно может быть жестким или упругим, с незначительной жесткостью на изгиб. Выводятся интегральные уравнения для разных случаев и описываются их решения. Исследуются напряженные состояния, вокруг сингулярных точек. Определяются пары факторов интенсивности напряжений, похожие на такие же для задач со щелью. Разработана серия численных примеров, для двух соединенных полуплоскостей и для полуплоскости. Факторы интенсивности напряжений являются функциями отношения расстоянив от границы раздела и длины щели.